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# Harmonic Relations between Green's Functions and Green's Matrices for Boundary Value Problems III

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## 1 Introduction

This is a continuation of previous papers [4], [6] and treats a two-point boundary value problem for the semilinear ODE

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+f(x,u)=0\ ,\ a<x<b \tag{1.1}$$

subject to separated boundary conditions

$$B_1(u)=\alpha_1u(a)-\alpha_2u'(a)=0\ ,\ (\alpha_1,\alpha_2)\neq(0,0), \tag{1.2}$$

$$B_2(u)=\beta_1u(b)+\beta_2u'(b)=0\ ,\ (\alpha_1,\alpha_2)\neq(0,0). \tag{1.3}$$

We assume that $\alpha_i\geqq 0,\ \beta_i\geqq 0,\ i=1,2,\ p\in C^1[a,b],\ p(x)>0$ in $[a,b],\ f\in C([a,b]\times\mathbb{R}),\ \frac{\partial f}{\partial u}$ exists, is continuous and nonnegative in $[a,b]\times\mathbb{R}$.
In order to discretize (1.1)–(1.3), we put

$$a=x_0<x_1<\cdots<x_n<x_{n+1}=b\ ,\ x_{i+\frac{1}{2}}=\frac{1}{2}(x_i+x_{i+1}),$$

$$h_i=x_i-x_{i-1}\ ,\ h=\max_i h_i.$$

Then the Shortley-Weller approximation at inner nodes $x_i$ , $1\leqq i\leqq n$ , is defined by

$$-\frac{p_{i+\frac{1}{2}}(U_{i+1}-U_i)/h_{i+1}-p_{i-\frac{1}{2}}(U_i-U_{i-1})/h_i}{(h_i+h_{i+1})/2}+f(x_i,U_i)=0,$$

or

$$\frac{2}{h_i+h_{i+1}}\left[-a_i^{(sw)}U_{i-1}+\left(a_i^{(sw)}+a_{i+1}^{(sw)}\right)U_i-a_{i+1}^{(sw)}U_{i+1}\right]+f(x_i,U_i)=0\ ,\ 1\leqq i\leqq n, \tag{1.4}$$

where $a_i^{(sw)}=p_{i-\frac{1}{2}}/h_i$ and $U_i$ denote approximations of exact values $u_i=u(x_i)$. Furthermore, the equation (1.1) is discretized at $x_0$ by

$$U_0=0\quad(\text{if }\alpha_2=0)$$

or

$$\frac{2}{h_i}\left[\left(a_0^{(sw)}+\widetilde{a}_1^{(sw)}\right)U_0-\widetilde{a}_1^{(sw)}U_1\right]+f(x_0,U_0)=0 \text{ (if } \alpha_2\neq 0), \tag{1.5}$$

where

$$a_0^{(sw)}=\frac{\alpha_1}{\alpha_2}p_0-\frac{\alpha_1 p_0{}'}{2\alpha_2}h_1 \quad\text{and}\quad \widetilde{a}_1^{(sw)}=\frac{p_0}{h_1}=a_1^{(sw)}+O(1)$$

Similarly, at $x_{n+1}$, we have

$$U_{n+1}=0 \quad (\text{if } \beta_2=0)$$

or

$$\frac{2}{h_{n+1}}\left[-\widetilde{a}_{n+1}^{(sw)}U_n+\left(\widetilde{a}_{n+1}^{(sw)}+a_{n+2}^{(sw)}\right)U_{n+1}\right]+f(x_{n+1},U_{n+1})=0 \text{ (if } \beta_2\neq 0), \tag{1.6}$$

where

$$\begin{aligned}\widetilde{a}_{n+1}^{(sw)}&=\frac{p_{n+1}}{h_{n+1}}=a_{n+1}^{(sw)}+O(1),\\ a_{n+2}^{(sw)}&=\frac{\beta_1}{\beta_2}p_{n+1}+\frac{\beta_1}{2\beta_2}p_{n+1}'h_{n+1}.\end{aligned}$$

The formulas (1.5) and (1.6) are obtained with the use of a ficticious node method.

Observe that if $\alpha_2\beta_2\neq 0$, then the above discretized system can be written in a matrix-vector form

$$HA^{(sw)}U+F(U)=0 \tag{1.7}$$

where

$$H=\operatorname{diag}\left(\frac{2}{h_1},\frac{2}{h_1+h_2},\cdots,\frac{2}{h_n+h_{n+1}},\frac{2}{h_{n+1}}\right),$$

$$A^{(sw)}=\begin{pmatrix} a_0^{(sw)}+\widetilde{a}_1^{(sw)} & -\widetilde{a}_1^{(sw)} & & & \\ -a_1^{(sw)} & a_1^{(sw)}+a_2^{(sw)} & -a_2^{(sw)} & & \\ \ddots & \ddots & \ddots & & \\ & -a_n^{(sw)} & a_n^{(sw)}+a_{n+1}^{(sw)} & -a_{n+1}^{(sw)} \\ & & -\widetilde{a}_{n+1}^{(sw)} & \widetilde{a}_{n+1}^{(sw)}+a_{n+2}^{(sw)} \end{pmatrix},$$

$$\begin{aligned}U&=(U_0,U_1,\cdots,U_{n+1})^t,\\ F(U)&=(f(x_0,U_0),\cdots,f(x_{n+1},U_{n+1}))^t.\end{aligned}$$

For the case $\alpha_2\beta_2=0$, we obtain equations similar to (1.7). For example, if the boundary conditions are of Dirichlet's type ($\alpha_2=\beta_2=0$ and $\alpha_1=\beta_1=1$), then (1.7) is replaced by

$$\widehat{H}\widehat{A}^{(sw)}\widehat{U}+\widehat{F}(\widehat{U})=0$$

with

$$\widehat{H} = \mathrm{diag}\left(\frac{2}{h_1+h_2}, \cdots, \frac{2}{h_n+h_{n+1}}\right),$$

$$\widehat{A}^{(sw)} = \begin{pmatrix} a_1^{(sw)}+a_2^{(sw)} & -a_2^{(sw)} & & & \\ -a_2^{(sw)} & a_2^{(sw)}+a_3^{(sw)} & -a_3^{(sw)} & & \\ \ddots & \ddots & \ddots & & \\ & -a_{n-1}^{(sw)} & a_{n-1}^{(sw)}+a_n^{(sw)} & -a_n^{(sw)} \\ & & -a_n^{(sw)} & a_n^{(sw)}+a_{n+1}^{(sw)} \end{pmatrix},$$

$$\widehat{U} = (U_1, \cdots, U_n)^t,$$

and

$$\widehat{F}(\widehat{U}) = (f(x_1, U_1), \cdots, f(x_n, U_n))^t$$

In this case, it is shown (cf. [1], [4]–[6]) that there is a harmonic relation between the Green function $\widehat{G}(x,\xi)$ for the operator $L : u \to -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)$ on $\widehat{\mathscr{D}} = \{u \in C^2[a,b] \mid u(a) = u(b) = 0\ ,\ i = 1,2\}$ and the Green matrix $[\widehat{A}^{(sw)}]^{-1} = (\widehat{g}_{ij}^{(sw)})$, which is also called the discrete Green function : That is, if $p \in C^{1,1}[a,b]$, then we have

$$\widehat{G}(x_i, x_j) - \widehat{g}_{ij}^{(sw)} = \begin{cases} O(h^3) & (j \in \Gamma) \\ O(h^2) & (j \notin \Gamma) \end{cases} \tag{1.8}$$

where $\Gamma = \{1, 2, \cdots, n_a, n-n_b+1, n-n_b+2, \cdots, n\}$ with arbitrarily given positive integers $n_a$ and $n_b \ll n$ which are fixed. It can also be shown that (1.8) holds between the Green function for the operator $\widehat{L}\ :\ u\ \to -\frac{d}{dx}(p(x)\frac{du}{dx}) + q(x)\frac{du}{dx} + r(x)u$ on $\widehat{\mathscr{D}}$ and the corresponding Green matrix, if $p \in C^{3,1}[a,b]\ ,\ q, r \in C^{1,1}[a,b]$ (cf. [6]).

On the basis of this result, we can prove that

$$u_i - U_i = \begin{cases} O(h^3) & (i \in \Gamma) \\ O(h^2) & (i \notin \Gamma)\ , \end{cases} \tag{1.9}$$

for the problem

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)\frac{du}{dx} + f(x,u) = 0\ ,\ u \in \widehat{\mathscr{D}}\ ,$$

provided that $p, u \in C^{3,1}$, $q \in C^{1,1}[a,b]$. For the general boundary conditions (1.2) and (1.3), however, the relations (1.8) and (1.9) do not hold in general.

Then, the following question arises :

*Let $G(x,\xi)$ be the Green function for the operator $L$ on $\mathscr{D} = \{u \in C^2[a,b] \mid B_i(u) = 0, i = 1,2\}$. Then, what is a matrix $A = (a_{ij})$ such that $A^{-1} = (G(x_i, x_j))$ ?*

The purpose of this paper is to give an answer to this question which leads to a new discretized system and to estimate the error of the numerical solution for the system. Existence theorems of solution for the continuous problem (1.1)–(1.3) and the corresponding discrete one are also given.

## 2 Green's function and Green's matrix

We keep the notation and the assumptions in §1 : $p \in C^1[a,b]$ , $p > 0$ , $f \in C^1([a,b] \times \mathbb{R})$ , $f_u \geqq 0$ , $\alpha_i \geqq 0$ , $\beta_i \geqq 0$ , $i = 1,2$ , $\alpha_1 + \alpha_2 > 0$ , and $\beta_1 + \beta_2 > 0$.

**Lemma 2.1** *The Green function $G(x,\xi)$ for $(L, \mathscr{D})$ exists if and only if $\alpha_1 + \beta_1 > 0$.*

*Proof.* Since $\varphi_1(x) = 1$ and $\varphi_2(x) = \int_a^b \frac{ds}{p(s)}$ are the fundamental solutions of $Lu = 0$ , $u \in \mathscr{D}$, the Green function exists if and only if

$$\Delta \equiv \begin{vmatrix} B_1(\varphi_1) & B_1(\varphi_2) \\ B_2(\varphi_1) & B_1(\varphi_2) \end{vmatrix} = \begin{vmatrix} \alpha_1 & -\dfrac{\alpha_2}{p(a)} \\ \beta_1 & \beta_1 \displaystyle\int_a^b \frac{ds}{p(s)} + \frac{\beta_2}{p(b)} \end{vmatrix} \neq 0.$$

It is clear that this condition is equivalent to $\alpha_1 + \beta_1 > 0$. Q.E.D.

**Lemma 2.2** *If $\alpha_1 + \beta_1 > 0$, then the Green function $G(x,\xi)$ is given by*

$$G(x,\xi) = \begin{cases} \dfrac{1}{\Delta}\left(\dfrac{\alpha_2}{p(a)} + \alpha_1 \displaystyle\int_a^x \frac{ds}{p(s)}\right)\left(\dfrac{\beta_2}{p(b)} + \beta_1 \displaystyle\int_\xi^b \frac{ds}{p(s)}\right) & (x \leqq \xi) \\ \dfrac{1}{\Delta}\left(\dfrac{\alpha_2}{p(a)} + \alpha_1 \displaystyle\int_a^\xi \frac{ds}{p(s)}\right)\left(\dfrac{\beta_2}{p(b)} + \beta_1 \displaystyle\int_x^b \frac{ds}{p(s)}\right) & (x \geqq \xi) \end{cases}$$

$$= \begin{cases} \dfrac{1}{\bar{\Delta}}\left(\dfrac{\alpha_2}{\alpha_1 p(a)} + \displaystyle\int_a^x \frac{ds}{p(s)}\right)\left(\dfrac{\beta_2}{\beta_1 p(b)} + \displaystyle\int_\xi^b \frac{ds}{p(s)}\right) & (x \leqq \xi) \\ \dfrac{1}{\bar{\Delta}}\left(\dfrac{\alpha_2}{\alpha_1 p(a)} + \displaystyle\int_a^\xi \frac{ds}{p(s)}\right)\left(\dfrac{\beta_2}{\beta_1 p(b)} + \displaystyle\int_x^b \frac{ds}{p(s)}\right) & (x \geqq \xi) \end{cases}$$

*(if $\alpha_1\beta_1 \neq 0$),*

*where*

$$\bar{\Delta} = \frac{\alpha_2}{\alpha_1 p(a)} + \frac{\beta_2}{\beta_1 p(b)} + \int_a^b \frac{ds}{p(s)}.$$

**Lemma 2.3** *If $\alpha_1\alpha_2\beta_1\beta_2 \neq 0$, then put*

$$a_i = \begin{cases} \dfrac{\alpha_1}{\alpha_2} p(a) & (i = 0) \\ 1 \Big/ \displaystyle\int_{x_{i-1}}^{x_i} \frac{ds}{p(s)} & (1 \leqq i \leqq n+1) \\ \dfrac{\beta_1}{\beta_2} p(b) & (i = n+2) \end{cases}$$

*and*

$$A = \begin{pmatrix} a_0 + a_1 & -a_1 & & & \\ -a_1 & a_1 + a_2 & -a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & -a_{n+1} & a_{n+1} + a_{n+2} \end{pmatrix}$$

*Then $A$ is an $M$-matrix and $A^{-1} = (G(x_i, x_j))$.*

*Proof.* This follows from Theorem 3.2 in [5] and Lemma 2.2. Q.E.D.

Lemma 2.3 leads to a new discretized system

$$HAU + F(U) = 0 \tag{2.1}$$

for solving (1.1)–(1.3), provided that $\alpha_1\alpha_2\beta_1\beta_2 \neq 0$. It is interesting to compare $A$ and $A^{-1}$ with $A^{(sw)}$ and $[A^{(sw)}]^{-1} = (g_{ij}^{(sw)})$, respectively.
We then have

$$\begin{aligned}
a_0^{(sw)} &= a_0 + O(h_1), \\
\widetilde{a}_1^{(sw)} &= a_1 + O(1), \\
\widetilde{a}_{n+1}^{(sw)} &= a_{n+1} + O(1), \\
a_{n+2}^{(sw)} &= a_{n+2} + O(h_n + 1)
\end{aligned}$$

and

$$a_i^{(sw)} = a_i + O(h_i^3)\ ,\ 1 \leqq i \leqq n+1\ (\text{if } p \in C^{1,1}[a,b]).$$

Furthermore, in order to compare $A^{-1}$ with $[A^{(sw)}]^{-1}$, let

$$d_0 = \frac{p_{\frac{1}{2}}}{p_0}\ ,\ d_{n+1} = \frac{p_{n+\frac{1}{2}}}{p_{n+1}}\ ,\ d_i = 1\ (1 \leqq i \leqq n)$$

and

$$D = \operatorname{diag}(d_0, d_1, \cdots, d_n, d_{n+1}).$$

Then

$$DA^{(sw)} = \begin{pmatrix} a_0^{*(sw)} + a_1^{(sw)} & -a_1^{(sw)} & & & \\ -a_1^{(sw)} & a_1^{(sw)} + a_2^{(sw)} & -a_2^{(sw)} & & \\ \ddots & \ddots & \ddots & & \\ & & & -a_{n+1}^{(sw)} & a_{n+1}^{(sw)} + a_{n+2}^{*(sw)} \end{pmatrix} = \widetilde{A}^{(sw)}\ (\text{say}),$$

where

$$\begin{aligned}
a_0^{*(sw)} &= \frac{p_{\frac{1}{2}}}{p_0}\left(\frac{\alpha_1}{\alpha_2}p_0 - \frac{\alpha_1 p_0'}{2\alpha_2}h_1\right) = \frac{\alpha_1}{\alpha_2}p_{\frac{1}{2}} - \frac{\alpha_1}{2\alpha_2}\frac{p_0'}{p_0}p_{\frac{1}{2}}h_1 \\
&= \frac{\alpha_1}{\alpha_2}p_0 + O(h_1^2) = a_0 + O(h_1^2)
\end{aligned}$$

and, similarly,

$$a_{n+2}^{*(sw)} = \frac{p_{n+\frac{1}{2}}}{p_{n+1}}\left(\frac{\beta_1}{\beta_2}p_{n+1} + \frac{\beta_1}{2\beta_2}p_{n+1}'h_{n+1}\right) = a_{n+2} + O(h_{n+1}^2).$$

Hence, by Theorem 3.2 in [5], the $(n+2)\times(n+2)$ Green matrix $[\widetilde{A}^{(sw)}]^{-1} = (\widetilde{g}_{ij}^{(sw)})$ , $0 \leqq i$ , $j \leqq n+1$ is given by

$$\widetilde{g}_{ij}^{(sw)} = \begin{cases} \dfrac{1}{z_{n+2}} z_i(z_{n+2} - z_j) & (i \leqq j) \\ \dfrac{1}{z_{n+2}} z_j(z_{n+2} - z_i) & (i \geqq j) \end{cases}$$

with

$$\begin{aligned} z_0 &= \frac{1}{a_0^{*(sw)}} = \frac{1}{a_0} + O(h_1^2), \\ z_i &= \frac{1}{a_0^{*(sw)}} + \sum_{k=1}^{i} \frac{1}{a_k^{(sw)}} \\ &= \frac{1}{a_0} + O(h_1^2) + \sum_{k=1}^{i} \frac{1}{a_k + O(h_i^3)} \\ &= \sum_{k=0}^{i} \frac{1}{a_k} + O(h^2) \\ &= \frac{\alpha_2}{\alpha_1 p(a)} + \int_a^{x_i} \frac{ds}{p(s)} + O(h^2) \quad (1 \leqq i \leqq n+1), \\ z_{n+2} &= \frac{\alpha_2}{\alpha_1 p(a)} + \frac{\beta_2}{\beta_1 p(b)} + \int_a^b \frac{ds}{p(s)} + O(h^2) \\ &= \overline{\Delta} + O(h^2). \end{aligned}$$

We thus obtain that if $p \in C^{1,1}[a,b]$, then

$$\widetilde{g}_{ij}^{(sw)} = G(x_i, x_j) + O(h^2) \quad \forall i, j.$$

Since

$$\left[A^{(sw)}\right]^{-1} = \left[\widetilde{A}^{(sw)}\right]^{-1} D \ , \ d_0 = 1 + O(h_1) \text{ and } d_{n+1} = O(h_{n+1}),$$

we have the following result.

**Theorem 2.1** *The Green matrix $[A^{(sw)}]^{-1}$ for (1.7) approximates the Green function $G(x,\xi)$ as follows :*

$$g_{ij}^{(sw)} = \begin{cases} G(x_i, x_j) + O(h^2) & (j \neq 0 \ , \ n+1) \\ G(x_i, x_j) + O(h) & (j = 0 \ , \ n+1). \end{cases}$$

## 3 Existence of solution

Before estimating errors of (2.1), we state existence theorems of solution for continuous problem (1.1)–(1.3) and the corresponding discretized system (2.1), since both equations are nonlinear so that the existence of solution is not trivial.

**Theorem 3.1** *If $\alpha_1 + \beta_1 > 0$, then the boundary value problem (1.1)–(1.3) has a unique solution in $\mathscr{D}$.*

*Proof.* (i)Uniqueness. Let $u$ and $v$ be two solutions of (1.1)–(1.3) in $\mathscr{D}$ and put $w = u - v$. Then $w \in \mathscr{D}$ and $w$ satisfies

$$-\frac{d}{dx}\left(p(x)\frac{dw}{dx}\right) + \left(\int_0^1 f_u(x, v + \theta w)d\theta\right) w = 0 \ , \ a < x < b. \tag{3.1}$$

Multiplying (3.1) by $w$ and integrating it from $a$ to $b$, we obtain

$$\int_a^b \left\{ p\left(\frac{dw}{dx}\right)^2 + \left(\int_0^1 f_u(x, v + \theta w)d\theta\right) w^2 \right\} dx + W = 0, \tag{3.2}$$

where

$$W = \left[-p\frac{dw}{dx}w\right]_a^b = \begin{cases} \dfrac{\alpha_1}{\alpha_2}p(a)w(a)^2 + \dfrac{\beta_1}{\beta_2}p(b)w(b)^2 & (\alpha_2\beta_2 \neq 0) \\ \dfrac{\alpha_1}{\alpha_2}p(a)w(a)^2 & (\alpha_2 \neq 0 \ , \ \beta_2 = 0) \\ \dfrac{\beta_1}{\beta_2}p(b)w(b)^2 & (\alpha_2 = 0 \ , \ \beta_2 \neq 0) \\ 0 & (\alpha_2 = \beta_2 = 0) \end{cases} \tag{3.3}$$

$$\geqq 0.$$

Hence we have from (3.2) $\frac{dw}{dx} = W = 0$. It follows from the expression (3.3) that the boundary conditions $B_i(w) = 0 \ , \ i = 1,2$ with $\alpha_1 + \alpha_2 > 0 \ , \ \beta_1 + \beta_2 > 0 \ , \ \alpha_1 + \beta_1 > 0$ and $W = 0$ imply $w(a) = 0$ or $w(b) = 0$. This, together with $\frac{dw}{dx} = 0$, yields $w = 0$.

(ii)Existence. Let $C[a,b]$ be a Banach space with the norm $\|u\|_\infty = \sup_{a \leqq x \leqq b} |u(x)|$ and

$$\Omega = \{u \in C[a,b] \mid \|u\|_\infty \leqq \gamma \equiv M(b-a)\|f_0\|_\infty\},$$

where

$$M = \max_{a \leqq x, \xi \leqq b} |G(x,\xi)| \ , \ f_0(x) = f(x,0).$$

Given a function $u \in \Omega$, define a linear operator $\widetilde{L} \ : \ \mathscr{D} \to C[a,b]$ by

$$\widetilde{L}v \equiv -\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + \int_0^1 f_u(x, \theta u)d\theta v \ , \ v \in \mathscr{D}.$$

Then the Green function $\widetilde{G}(x,\xi)$ for $(\widetilde{L}, \mathscr{D})$ exists and the solution of the equation $\widetilde{L}v = -f_0(x) \ , \ v \in \mathscr{D}$ can be written

$$v(x) = -\int_a^b \widetilde{G}(x,\xi)f_0(\xi)d\xi.$$

Then $v \in \Omega$, since, as is well known, $0 \leqq \widetilde{G}(x,\xi) \leqq G(x,\xi)$ and $\|v\|_\infty \leqq \int_a^b G(x,\xi)\|f_0\|_\infty d\xi \leqq \gamma$. Hence we can define an operator $T : \Omega \to \Omega$ by $Tu = v$. It can then be shown that $\overline{T(\Omega)}$ is compact in $C[a,b]$ by Ascoli-Arzela's theorem. Hence, by the Schauder fixed point theorem, $T$ has a fixed point $u \in \Omega \cap \mathscr{D}$, which is a solution of (1.1)–(1.3). Q.E.D.

**Remark 3.1** Theorem 3.1 is not included in Keller's result [1:Theorem1.2.2], since his theorem requires $f_u > 0$.

**Theorem 3.2** *The system (2.1) has a unique solution for any nodes.*

*Proof.* This follows from the following result which may be found in Ortega-Rheinboldt [3] :

If $A$ is an $n \times n$ $M$-matrix and $f(x,u)$ is monotonically increasing with respect to $u$, then the equation

$$\begin{aligned} &AU + (f(x_1,U_1),\cdots,f(x_n,U_n))^t = 0, \\ &U = (U_1,\cdots,U_n)^t \end{aligned}$$

has a unique solution. Q.E.D.

**Remark 3.2** This result applies to (1.7), too.

## 4 Error Estimates

In this section, we shall show that the solution $U$ of (2.1) has the second-order accuracy. By Lemma 2.3 and (2.1), we have

$$U_i + \sum_{j=0}^{n+1} G(x_i,x_j)w_j f(x_j,U_j) = 0 \tag{4.1}$$

and

$$u_i + \int_a^b G(x_i,\xi) f(\xi,u(\xi))\,d\xi = 0, \tag{4.2}$$

where

$$w_j = \begin{cases} \dfrac{h_1}{2} & (j=0) \\ \dfrac{h_j + h_{j+1}}{2} & (j=1,2,\cdots,n) \\ \dfrac{h_{n+1}}{2} & (j=n+1). \end{cases}$$

If $p \in C^1[a,b]$, then $G(x_i,x_j)$ belongs to $C^2$ class for each subintervals $[x_k,x_{k+1}]$ , $0 \leqq k \leqq n$. Hence, if $\frac{d}{dx}(p\frac{du}{dx}) \in C^{1,1}[a,b]$, then $f(x,u(x)) \in C^{1,1}[a,b]$ and

$$\int_a^b G(x_i,\xi) f(\xi,u(\xi))\,d\xi = \sum_{j=0}^{n+1} G(x_i,x_j)w_j f(x_j,u(x_j)) + O(h^2)$$

by the well known fact for the error of trapezoidal rule in numerical integration. It now follows from (4.1) and (4.2) that

$$u_i - U_i + \sum_{j=0}^{n+1} G(x_i, x_j) w_j \left(f(x_j, u_j) - f(x_j, U_j)\right) = O(h^2)$$

or

$$(HA + D)(\boldsymbol{u} - \boldsymbol{U}) = HAO(h^2), \tag{4.3}$$

where

$$D = \operatorname{diag}\left(f_u(x_0, \eta_0), \cdots, f_u(x_{n+1}, \eta_{n+1})\right)$$

with

$$\eta_i = U_i + \theta_i(u_i - U_i)\ ,\ 0 < \theta_i < 1\ ,\ 0 \leqq i \leqq n+1.$$

Since $f_u \geqq 0$, the diagonal matrix $D$ is nonnegative and $HA + D$ is an $M$-matrix. We thus obtain from (4.3)

$$\begin{aligned} \boldsymbol{u} - \boldsymbol{U} &= (HA + D)^{-1} HAO(h^2) \\ &= \left[(HA)^{-1} - (HA + D)^{-1} D (HA)^{-1}\right] HAO(h^2) \\ &= O(h^2) - (HA + D)^{-1} DO(h^2). \end{aligned} \tag{4.4}$$

The usual convergence theory for the finite difference method tells us that

$$u_i - U_i = O\left(\max_j |\tau_j|\right),\ \forall i,$$

where $\tau_j$ stands for the local trunction error of (2.1) at $x_j$. Furthermore, as is easily seen, if $p \in C^1[a, b]$ and $u \in C^2[a, b]$, then $\tau_j \to 0$ as $h \to 0$. Therefore, $U_i \to u_i$ as $h \to 0$ so that $D$ is bounded and it is easy to see that

$$\begin{aligned} \left|(HA + D)^{-1} DO(h^2)\right| &\leqq (HA + D)^{-1} \left|DO(h^2)\right| \\ &= (HA + D)^{-1} \left|O(h^2)\right| \\ &\leqq (HA)^{-1} \left|O(h^2)\right| \\ &= O(h^2) \end{aligned}$$

where we have used the notation $|\boldsymbol{V}| = (|V_0|, \cdots, |V_{n+1}|)^t$ for $\boldsymbol{V} = (V_0, \cdots, V_{n+1})^t$. Consequently we obtain

$$u_i - U_i = O(h^2) \quad \forall i$$

for any nodes $\{x_i\}$, under the assumption $\frac{d}{dx}(p\frac{du}{dx}) \in C^{1,1}[a, b]$, which is satisfied if $p \in C^{2,1}[a, b]$ and $u \in C^{3,1}[a, b]$ . We state this as

**Theorem 4.1** *If $p \in C^{2,1}[a,b]$ and $u \in C^{3,1}[a,b]$, then the discretized system (2.1) has the second order accuracy for any nodes.*

**Theorem 4.2** *Under the same assumptions as in Theorem 4.1, the Shortley-Weller approximation (1.7) has the second order accuracy for any nodes.*

*Proof.* The same proof as in [6] works by using Theorem 2.1.

**Remark 4.1** We can extend the argument developed in this paper to the boundary value problem

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+f\left(x,u,\frac{du}{dx}\right)=0\ ,\ a<x<b$$
$$B_i(u)=0\ ,\ i=1,2,$$

where $f \in C([a,b]\times\mathbb{R}^2)$ , $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial u'}$ exist in $[a,b]\times\mathbb{R}^2$ , $\frac{\partial f}{\partial u} \geqq 0$ and $\frac{\partial f}{\partial u'}$ is bounded. This will be discussed elsewhere.

## References


[1] Q.Fang, T.Tsuchiya and T.Yamamoto, Finite difference, finite element and finite volume methods for two-point boundary value problems, J.Comp.Appl.Math.139(2002), 9-19

[2] H.B.Keller, Numerical Methods for Two-Point Boundary Value Problems, Blaisdell, 1968.

[3] J.Ortega and W.Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, 1970 (Reprinted by SIAM, 2000).

[4] T.Yamamoto, Harmonic relations between Green's functions and Green's matrices for boundary value problems (in RIMS Kokyuroku 1169, RIMS, Kyoto University, 2000), 15-26.

[5] T.Yamamoto, Inversion formulas for tridiagonal matrices with applications to boundary value problems, Numer. Funct. Anal. & Optimiz. 22(2001), 357-385.

[6] T.Yamamoto, Harmonic relations between Green's functions and Green's matrices for boundary value problems II (in RIMS Kokyuroku 1286, RIMS, Kyoto University, 2002), 27-33.